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On approximations of the de Rham complex and their cohomology

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Abstract

For a commutative algebra R, its de Rham cohomology is an important invariant of R. In the paper, an infinite chain of de Rham-like complexes is introduced where the first member of the chain is the de Rham complex. The complexes are called *approximations of the de Rham complex.* Their cohomologies are found for polynomial rings and algebras of power series over a field of characteristic zero.

Key Words: differentials, the de Rham complex, the de Rham cohomology, polynomial algebra, algebra of power series, approximations.

Mathematics subject classification 2010: 13D03, 13N05, 13N10, 13N15.

1 Introduction

Let R be a commutative K-algebra with 1 over a commutative ring K. Module means a left module. For each natural number $m \geq 1$, let $\Omega_m(R)$ be the universal module of derivations of order m of R (the module of m^{th} order differentials), see [7, 4, 6] and Section 2. The modules $\Omega_m(R)$ were studied in [5, 7, 4, 6, 1, 8, 2, 3] to name just a few. In particular, Ω_1 is the module of differentials of R over K and the exterior algebra of the left R-module Ω_1, $(\wedge^\bullet \Omega_1, d_1)$, is the de Rham cochain complex of R. There is a chain of cochain complexes

$$\cdots \to \wedge^\bullet \Omega_m \to \cdots \to \wedge^\bullet \Omega_2 \to \wedge^\bullet \Omega_1 \to 0$$

(see (21)) that are called *approximations of the de Rham complex.* The main result of the paper is an explicit description of the cohomology groups $H^\bullet(R, m) := H^\bullet(\wedge^\bullet \Omega_m)$ for the polynomial algebra $P_n = K[x_1, \dots, x_n]$ and the algebra $S_n = K[[x_1, \dots, x_n]]$ of power series over a field K of characteristic zero (below $\binom{i}{j} = \frac{i!}{j!(i-j)!}$ is the binomial coefficient):

- (Theorem 2.7)

$$H^i(P_n, m) \simeq \begin{cases} K^{\binom{\mathrm{rk}(\Omega_m)-n}{i}} & \textit{if } 0 \leq i \leq \mathrm{rk}(\Omega_m) - n, \\ 0 & \textit{otherwise}, \end{cases}$$

where $\mathrm{rk}(\Omega_m)=\binom{n+m}{n}-1$.

- (Theorem 3.2)

$$H^i(S_n,m)\simeq\begin{cases}K^{\binom{\mathrm{rk}(\Omega_m)-n}{i}} & \textit{if } 0\leq i\leq \mathrm{rk}(\Omega_m)-n,\\ 0 & \textit{otherwise,}\end{cases}$$

where $\mathrm{rk}(\Omega_m)= \binom{n+m}{n}-1$.

2 Approximations of the de Rham complex

In this paper, a module means a left module. Let R be a commutative K-algebra where K is a commutative ring, $R\otimes R:=R\otimes_K R$, $E:=\mathrm{End}_K(R\otimes R)$ be the endomorphism algebra of $R\otimes R$, i.e., the algebra of all K-homomorphisms $R\otimes R\to R\otimes R$. Let M be an R-bimodule. A K-linear map $\partial:R\to M$ such that $\partial(rs)=\partial(r)s+r\partial(s)$ is called a K-*derivation* from R to M. The set of all K-derivations from R to M is denoted by $\mathrm{Der}_K(R,M)$. In particular, for $M=R$, $\mathrm{Der}_K(R):=\mathrm{Der}_K(R,R)$ is the set of all K-*derivations* of the K-algebra R. For each $a\in R$, let

$$\ell_a: R\otimes R\to R\otimes R,\ \ b\otimes c\mapsto ab\otimes c, \tag{1}$$

$$\tau_a: R\otimes R\to R\otimes R,\ \ b\otimes c\mapsto b\otimes ca. \tag{2}$$

The maps ℓ_a,τ_a and $\Delta_a:=\ell_a-\tau_a$ commute. The algebra $R\otimes R$ contains two subalgebras $R\otimes 1$ and $1\otimes R$. The map

$$d:R\to R\otimes R,\ \ r\mapsto d(r):=r':=r\otimes 1-1\otimes r \tag{3}$$

is a K-derivation, $d\in\mathrm{Der}_K(R,R\otimes R)$, that is $(rs)^{'}=r^{'}s+rs^{'}=r'\cdot 1\otimes s+r\otimes 1\cdot s'$ for all $r,s\in R$. Let I be the kernel of the algebra epimorphism

$$\varphi: R\otimes R\to R,\ \ r\otimes s\mapsto rs. \tag{4}$$

Then φd=0, so $R^{'}$:=dR:=im$(d)\subseteq I$ and the map

$$d:R\to I,\ \ r\mapsto r^{'}=r\otimes 1-1\otimes r \tag{5}$$

is a K-derivation, $d\in\mathrm{Der}_K(R,I)$.

Lemma 2.1 *1. $I=RR^{'}=R^{'}R$, i.e., the ideal I is generated by the set $R^{'}$ as a left or right R-module.*

2. $I^m=R(R^{'})^m=(R^{'})^mR$ for all $m\geq 1$.

Proof. 1. Statement 1 follows from the equality $r^{'}s=(rs)^{'}-rs^{'}$.
2. Statement 2 follows from statement 1. $\square$

The involution o. An automorphism of an algebra of degree 2 is called an *involution*. The map

$$o : R \otimes R \to R \otimes R, \quad r \otimes s \mapsto s \otimes r \tag{6}$$

is an involution since $(r\otimes s)^{oo} = r \otimes s$. Clearly, $(R\otimes 1)^o = 1\otimes R$ and $(1\otimes R)^o = R \otimes 1$. For all $r \in R$,

$$(r^{'})^o = -r^{'}. \tag{7}$$

Therefore, $I^o = I$, by Lemma 2.1. Let $x_1, x_2 \in R$. In particular, $x_1x_2 = x_2x_1$. Then

$$\begin{array}{lcl} x_1^{'}x_2^{'} & = & x_2^{'}x_1^{'} = x_1x_2^{'} - x_2^{'}x_1 = x_2x_1^{'} - x_1^{'}x_2, \\ (x_1x_2)^{'} & = & x_1^{'}x_2 + x_1x_2^{'} = x_2x_1^{'} - x_2x_1^{'} + x_1^{'}x_2 + x_1x_2^{'} = x_2x_1^{'} - x_1^{'}x_2^{'} + x_1x_2^{'}, \\ (x_1x_2)^{'} & = & x_1^{'}x_2 + x_1x_2^{'} = x_1^{'}x_2 + x_1x_2^{'} - x_2^{'}x_1 + x_2^{'}x_1 = x_1^{'}x_2 + x_1^{'}x_2^{'} + x_2^{'}x_1. \end{array}$$

The equalities above do not hold if the elements x_1 and x_2 do not commute. Let n be a natural number such that $n \geq 2$ and $[n] := \{1, \dots, n\}$. For a subset I of the set $[n]$, let $CI{:=}[n]\backslash I$ be its complement and $|I|$ be the number of elements in I.

Lemma 2.2 *Given elements $x_1, ..., x_n \in R$, we have*

$$(x_1 \cdots x_n)^{'} = \sum_{\phi \neq I \subseteq [n]} (-1)^{|I|+1} x^{CI} (x^{'})^I = \sum_{\phi \neq I \subseteq [n]} (x^{'})^I x^{CI} \tag{8}$$

where $x^{CI} := \prod_{j \in CI} x_j$ and $(x^{'})^I := \prod_{i\in I} x_i^{'}$. In particular,

$$(x_i^n)' = \sum_{m=1}^{n} (-1)^{m+1} \binom{n}{m} x_i^{n-m} x_i'^m = \sum_{m=1}^{n} \binom{n}{m} x_i'^m x_i^{n-m}.$$

More generally, for all $0 \neq \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$,

$$(x^\alpha)^{'} = \sum_{0 \neq \beta \leq \alpha} (-1)^{|\beta|+1} \binom{\alpha}{\beta} x^{\alpha - \beta} x'^\beta = \sum_{0 \neq \beta \leq \alpha} \binom{\alpha}{\beta} x'^\beta x^{\alpha-\beta} \tag{9}$$

where $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$, $x'^\beta := \prod_{i=1}^n x_i'^{\beta_i}$, $|\beta| := \beta_1 + \cdots + \beta_n$, $\beta \leq \alpha$ means $\beta_1 \leq \alpha_1, \dots, \beta_n \leq \alpha_n$, and $\binom{\alpha}{\beta} := \prod_{i=1}^n \binom{\alpha_i}{\beta_i}$ is a multi-nomial *coefficient. Furthermore, for a polynomial $P = P(t_1, \dots, t_n) \in K[t_1, \dots, t_n]$, let $p = P(x_1, \dots, x_n)$. Then*

$$p' = \sum_{\beta \neq 0} (-1)^{|\beta|+1} \frac{\partial^\beta p}{\partial x^\beta} \frac{x'^\beta}{\beta!} = \sum_{\beta \neq 0} \frac{x'^\beta}{\beta!} \frac{\partial^\beta p}{\partial x^\beta} \tag{10}$$

where $\frac{\partial^\beta p}{\partial x^\beta} = \frac{\partial^\beta p}{\partial t^\beta}|_{t_1=x_1,\dots,t_n=x_n}$.

Proof. Let us prove by induction on n that the second equality in (8) holds, i.e.,

$$(x_1 \cdots x_n)^{'} = \sum_{\phi \neq I \subseteq [n]} x'^{I} x^{CI}.$$

The case $n = 2$ was proven above. So, let $n > 2$ and we assume that the equality holds for all $n^{'} < n$. Now,

$$\begin{aligned} (x_1 \cdots x_n)^{'} &= (x_1 \cdots x_{n-1})^{'} x_n + x_1 \cdots x_{n-1} x_n' \\ &= \textstyle\sum_{\phi \neq J \subseteq [n-1]} x'^{J} x^{CJ} x_n + x_n^{'} x_1 \cdots x_{n-1} + x_1 \cdots x_{n-1} x_n' - x_n^{'} x_1 \cdots x_{n-1}. \end{aligned}$$

Notice that

$$x_1 \cdots x_{n-1} x_n' - x_n' x_1 \cdots x_{n-1} = (x_1 \cdots x_{n-1})^{'} x_n^{'} = \sum_{\phi \neq J \subseteq [n-1]} x'^{J} x^{CJ} x_n^{'}$$

and the second equality follows. By applying the automorphism o to the second equality we obtain the first equality:

$$(x_1 \cdots x_n)' = -((x_1 \cdots x_n)')^{o} = -\sum_{\phi \neq I \subseteq [n]} x^{CI} ((x')^{I})^{o} = \sum_{\phi \neq I \subseteq [n]} (-1)^{|I|+1} x^{CI} x'^{I},$$

by (7). The equalities in (9) follows from (8). The equality in (10) follows at once from (9). $\square$

The short exact sequence of left R-modules

$$0 \to I \to R \otimes R \stackrel{\varphi}{\to} R \to 0 \tag{11}$$

admits a section $\ell : R \to R \otimes R, \ \ r \mapsto r \otimes 1$, that is $\varphi \ell = \mathrm{id_R}$. Therefore,

$$R \otimes R = R \otimes 1 \oplus I \tag{12}$$

is the direct sum of left R-modules. Similarly, the short exact sequence of right R-modules (11) admits a section $r : R \to R \otimes R, \ \ a \mapsto 1 \otimes a$, that is $\varphi r = \mathrm{id_R}$. Therefore,

$$R \otimes R = 1 \otimes R \oplus I \tag{13}$$

is the direct sum of right R-modules. The I-*adic filtration* of the ring $R \otimes R$,

$$R \otimes R \supseteq I \supseteq I^2 \supseteq \cdots \supseteq I^m \supseteq \cdots$$

determines the chain of ring epimorphisms

$$\cdots \to R \otimes R/I^m \to \cdots \to R \otimes R/I^2 \to R \otimes R/I \to 0.$$

Let $\mathcal{P}(R):=\varprojlim R\otimes R/I^m$. For each $m\geq 1$, the ideal $\Omega_m := I/I^{m+1}$ of the ring $R\otimes R/I^{m+1}$ is called *the module of differentials of order* m of R. For all $m\geq 1$, by (12) and (13),

$$R\otimes R/I^{m+1} = R\otimes 1\oplus \Omega_m = 1\otimes R\oplus\Omega_m. \tag{14}$$

Let $\Omega_\infty := \varprojlim \Omega_m$ be the projective limit of $R\otimes R$-module epimorphisms

$$\cdots\to\Omega_m\to\cdots\to\Omega_2\to\Omega_1\to 0. \tag{15}$$

Then

$$\mathcal{P}(R) = R\otimes 1\oplus\Omega_\infty = 1\otimes R\oplus\Omega_\infty. \tag{16}$$

Clearly, Ω_∞ is an ideal of the ring $\mathcal{P}(R)$ such that $\mathcal{P}(R)/\Omega_\infty\simeq R$. For each $m\geq 1$, the derivation $d: R\to R\otimes R$ (see (3)) determines the derivation

$$d_m : R\to R\otimes R/I^{m+1}, \;\; r\mapsto r^{'}+I^{m+1}$$

which can be seen as *m'th approximation of the derivation d*. Recall that

$$R\otimes R/I^{m+1} = R\otimes 1\oplus\Omega_m = 1\otimes R\oplus\Omega_m.$$

By Lemma 2.1, $\mathrm{im}(d_m)\subseteq\Omega_m$. Therefore,

$$d_m : R\to\Omega_m, \;\; r\mapsto r^{'}+I^{m+1}$$

is a derivation of R-bimodules, i.e., $d_m(rs) = d_m(r)s+rd_m(s)$ for all elements $r,s\in R$. The commutative diagram

$$\begin{tikzcd}
 & R \arrow[d, "d_m"] \arrow[rrr, "d_2"] \arrow[rrrr, "d_1"] & & & & & \\
\ldots \arrow[r] & \Omega_m \arrow[r] & \ldots \arrow[r] & \Omega_2 \arrow[r] & \Omega_1 \arrow[r] & 0
\end{tikzcd}$$

yields the derivation

$$d_\infty : R\to\Omega_\infty.$$

The polynomial algebra case. Let $R=P_n := K[x_1,\ldots,x_n]$ be a polynomial algebra in variables $x_1,\ldots,x_n$ over a field K. The polynomial algebra $P_n\otimes P_n$ in $2n$ variables over K can be presented as the following polynomial algebras:

$$P_n\otimes 1[x_1',\ldots,x_n'] := P_n[x_1',\ldots,x_n'] := \{\textstyle\sum_{\alpha\in\mathbb{N}^n}\lambda_\alpha x'^\alpha \,|\, \lambda_\alpha\in P_n\otimes 1\} \text{ and}$$
$$1\otimes P_n[x_1',\ldots,x_n'] := [x_1',\ldots,x_n']P_n := \{\textstyle\sum_{\alpha\in\mathbb{N}^n}x'^\alpha\lambda_\alpha \,|\, \lambda_\alpha\in 1\otimes P_n\}$$

where $x_i' = x_i\otimes 1 - 1\otimes x_i$ and $x'^\alpha := x_1'^{\alpha_1}\cdots x_n'^{\alpha_n}$.

Proposition 2.3 *Let $R=P_n:=K[x_1,\ldots,x_n]$ be a polynomial algebra over a field K. Then*

1. $I = P_nP_n^{'} = \oplus_{|\alpha|\geq 1}P_nx'^{\alpha} = P_n'P_n = \oplus_{|\alpha|\geq 1}x'^{\alpha}P_n$ *where* $\alpha \in \mathbb{N}^n$ *and* $|\alpha| = \alpha_1 + \cdots + \alpha_n$. *For* $m \geq 1$, $I^m = \oplus_{|\alpha|\geq m}P_nx'^{\alpha} = \oplus_{|\alpha|\geq m}x'^{\alpha}P_n$. *The ideal* I *of* $P_n \otimes P_n$ *is equal to* $(x_1', \ldots, x_n')$.
2. *For* $m \geq 1$,

$$\Omega_m = I/I^{m+1} = \bigoplus_{1\leq|\alpha|\leq m} P_nx'^{\alpha} = \bigoplus_{1\leq|\alpha|\leq m} x'^{\alpha}P_n. \tag{17}$$

   In particular, the free left/right P_n*-module* Ω_m *has rank* $rk(\Omega_m) = \binom{n+m}{n} - 1$.
3. $\mathcal{P}(P_n) = P_n[[x_1^{'}, \ldots, x_n^{'}]] = [[x_1^{'}, \ldots, x_n^{'}]]P_n$ *is the algebra of power series with coefficients in the polynomial algebra* P_n *and*

$$\Omega_\infty = (x_1^{'}, \ldots, x_n^{'}) = \sum_{i=1}^n \mathcal{P}(P_n)x_i^{'} = \sum_{i=1}^n x_i^{'}\mathcal{P}(P_n)$$

   is the ideal of the algebra $\mathcal{P}(P_n)$ *generated by the elements* $x_1^{'}, \ldots, x_n^{'}$. *The derivation* $d_\infty : R \to \Omega_\infty$ *is given by (9).*
4. *For all* $m \geq 1$,

$$\Omega_m = \Omega_\infty/\Omega_\infty^{m+1}. \tag{18}$$

Proof. 1. By Lemma 2.1 and Lemma 2.2, $I = P_nP_n^{'} = \sum_{|\alpha|\geq 1} P_n(x^\alpha)' = \oplus_{|\alpha|\geq 1}P_nx'^{\alpha}$ and $I = P_n^{'}P_n = \sum_{|\alpha|\geq 1}(x^\alpha)'P_n = \oplus_{|\alpha|\geq 1}x'^{\alpha}P_n$ since $(x')^\alpha = x^\alpha \otimes 1 + \cdots + (-1)^{|\alpha|}1 \otimes x^\alpha$. Hence,

$$I^m = \bigoplus_{|\alpha|\geq m} P_nx'^{\alpha} = \bigoplus_{|\alpha|\geq m} x'^{\alpha}P_n$$

for all $m \geq 1$. Clearly, the ideal I of the algebra $P_n \otimes P_n$ is generated by the elements $x_1', \ldots, x_n'$.
2. Step 2 follows from statement 1.
3. Step 3 follows from statement 2.
4. Step 4 follows from statement 3. $\square$

Approximations of the de Rham complex. Let R be a commutative K-algebra. For each $m \geq 1$, let

$$\Lambda^\bullet\Omega_m = R \oplus \Omega_m \oplus \Lambda^2\Omega_m \oplus \cdots \oplus \Lambda^i\Omega_m \oplus \cdots$$

be the exterior algebra of the *left* R-module Ω_m. For each $i \geq 1$, the derivation $d_m = d_{m,0} : R \to \Omega_m$ can be extended to a map

$$d_{m,i} : \Lambda^i\Omega_m \to \Lambda^{i+1}\Omega_m, \;\; a_0a_1^{'} \wedge \cdots \wedge a_i^{'} \mapsto a_0^{'} \wedge a_1^{'} \wedge \cdots \wedge a_i^{'}.$$

$$R \xrightarrow{d_m = d_{m,0}} \Omega_m \xrightarrow{d_{m,1}} \Lambda^2\Omega_m \xrightarrow{d_{m,2}} \cdots \xrightarrow{d_{m,i-1}} \Lambda^i\Omega_m \xrightarrow{d_{m,i}} \cdots . \tag{19}$$

Lemma 2.4 *The complex (19) is a cochain complex, i.e.,* $d_{m,i+1}d_{m,i} = 0$ *for all* $i \geq 0$.

Proof. $d_{m,i+1}d_{m,i}(a_0a_1^{'} \wedge \cdots \wedge a_i^{'}) = d_{m,i+1}(a_0^{'} \wedge a_1^{'} \wedge \cdots \wedge a_i^{'}) = 1^{'} \wedge a_0^{'} \wedge a_1^{'} \wedge \cdots \wedge a_i^{'} = 0$, since $1' = 0$. Here, $a_i^{'} = d_m(a_i)$ where $d_m : R \to \Omega_m(R)$ denotes the universal derivation, see above. □

In a similar way, for each $m \geq 1$, the exterior algebra of the *right* R-module Ω_m is defined

$$\Lambda_r^{\bullet}\Omega_m = R \oplus \Omega_m \oplus \Lambda_r^2\Omega_m \oplus \cdots \oplus \Lambda_r^i\Omega_m \oplus \cdots.$$

We add the subscript 'r' to indicate that the right R-module structure is used for Ω_m. For each $i \geq 1$, the derivation

$$d_m = d_{m,0} = d_m^r : R \to \Omega_m$$

can be extended to a map

$$d_{m,i}^r : \Lambda_r^i\Omega_m \to \Lambda_r^{i+1}\Omega_m, \;\; a_1^{'} \wedge \cdots \wedge a_i^{'}a_{i+1} \mapsto a_1^{'} \wedge \cdots \wedge a_i^{'} \wedge a_{i+1}^{'}.$$

We have a cochain complex

$$R \xrightarrow{d_m = d_{m,0}^r} \Omega_m \xrightarrow{d_{m,1}^r} \Lambda^2\Omega_m \xrightarrow{d_{m,2}^r} \cdots \xrightarrow{d_{m,i-1}^r} \Lambda^i\Omega_m \xrightarrow{d_{m,i}^r} \cdots, \qquad (20)$$

$d_{m,i+1}^r d_{m,i}^r = 0$ for all $i \geq 0$. Clearly, the cochain complexes $(\wedge_r^{\bullet}\Omega_m, d_{m,i}^r)$ and $(\wedge_r^{\bullet}\Omega_m, (-1)^{i+1}d_{m,i}^r)$ have the same cohomology. The involution o of the ring $R \otimes R$ interchanges the left and right R-module structures of $R \otimes R$ (since, $(r \otimes 1)^o = 1 \otimes r$ for all $r \in R$). Hence, the involution $o : \Omega_m \to \Omega_m$, $a' \mapsto (a')^o = -a'$ interchanges the left and right R-module structures on Ω_m: for all $r, a \in R$,

$$(ra')^o = ((r \otimes 1)a')^o = (r \otimes 1)^o(a')^o = (1 \otimes r)(a')^o = (a')^o(1 \otimes r) = (a')^o r.$$

By the very definition, the exterior algebra $\wedge^{\bullet}\Omega_m$ (resp., $\wedge_r^{\bullet}\Omega_m$) of the left (resp., right) R-module Ω_m is an R-algebra where $R = R \otimes 1$ (resp., $R = 1 \otimes R$).

Lemma 2.5 *For each* $m \geq 1$*, the map*

$$o : \wedge^{\bullet}\Omega_m \to \wedge_r^{\bullet}\Omega_m, \;\; ra_1' \wedge \cdots \wedge a_i' \mapsto (ra_1' \wedge \cdots \wedge a_i')^o := r^o(a_1')^o \wedge \cdots \wedge (a_i')^o$$

is an isomorphism of R*-algebras, it is also an isomorphism of cochain complexes* $(\wedge^{\bullet}\Omega_m, d_{m,i})$ *and* $(\wedge_r^{\bullet}\Omega_m, (-1)^{i+1}d_{m,i}^r)$*. In particular, the cohomology of the three cochain complexes* $(\wedge^{\bullet}\Omega_m, d_{m,i})$*,* $(\wedge_r^{\bullet}\Omega_m, (-1)^{i+1}d_{m,i}^r)$ *and* $(\wedge_r^{\bullet}\Omega_m, d_{m,i}^r)$ *coincide.*

Proof. By the definition, the map $o : \wedge^{\bullet}\Omega_m \to \wedge_r^{\bullet}\Omega_m$ is an isomorphism of R-modules since (by (7))

$$(ra_1' \wedge \cdots \wedge a_i')^o = r^o(a_1')^o \wedge \cdots \wedge (a_i')^o = a_1' \wedge \cdots \wedge a_i'(-1)^i r = (a_1' \wedge \cdots \wedge a_i')^o r.$$

Furthermore,

$$\begin{array}{rcl} d^r_{m,i}((ra'_1\wedge\cdots\wedge a'_i)^o) & = & a'_1\wedge\cdots\wedge a'_i\wedge r'(-1)^i,\\ (d_{m,i}(ra'_1\wedge\cdots\wedge a'_i))^o & = & (r'\wedge a'_1\wedge\cdots\wedge a'_i)^o\\ & = & (-1)^{i+1}r'\wedge a'_1\wedge\cdots\wedge a'_i=-a'_1\wedge\cdots\wedge a'_i\wedge r', \end{array}$$

that is $((-1)^{i+1}d^r_{m,i})o = od_{m,i}$ and the map o yields an isomorphism of the cochain complexes $(\wedge^\bullet\Omega_m, d_{m,i})$ and $(\wedge^\bullet_r\Omega_m, (-1)^{i+1}d^r_{m,i})$. Now, the last statement of the lemma follows. $\square$

Definition 2.6 *For each natural number $m\geq 1$, let $H^\bullet(R,m)=\{H^i(R,m)\}_{i\geq 0}$ be the cohomology groups of the cochain complex (19).*

When $m=1$, the complex (19) is called *the de Rham complex of R* and its cohomology $H^\bullet_{DR}(R)$ is called *the de Rham cohomology of R*. The chain (15) yields the chain

$$\cdots\to\Lambda^\bullet\Omega_m\to\cdots\to\Lambda^\bullet\Omega_2\to\Lambda^\bullet\Omega_1\to 0 \tag{21}$$

of complexes that are called *approximations of the de Rham complex* and its projective limit $\varprojlim\Lambda^\bullet\Omega_m$ is a complex such that

$$(\varprojlim\Lambda^\bullet\Omega_m)_i=\varprojlim\Lambda^i\Omega_m. \tag{22}$$

The chain (21) yields the chain

$$\cdots\to H^\bullet(R,m)\to H^\bullet(R,m-1)\to\cdots\to H^\bullet(R,1)=H^\bullet_{DR}(R)\to 0. \tag{23}$$

In particular, for all $s\geq 0$, we have the chain

$$\cdots\to H^s(R,m)\to H^s(R,m-1)\to\cdots\to H^s(R,1)=H^s_{DR}(R)\to 0. \tag{24}$$

Let $\varprojlim_m H^\bullet(R,m)$ and $\varprojlim_m H^s(R,m)$ be the projective limits of (23) and (24), respectively. For natural numbers $n\geq 1$ and $m\geq 1$, let

$$\mathcal{H}_n(m):=\{\alpha\in\mathbb{N}^n\mid 1\leq|\alpha|\leq m\}\text{ where }|\alpha|:=\alpha_1+\cdots+\alpha_n.$$

Clearly,

$$|\mathcal{H}_n(m)|=\binom{n+m}{n}-1.$$

The degree Deg **and the associative filtration on $\wedge^s\Omega_m$.** For each $s=1,\dots,|\mathcal{H}_n(m)|$, $\wedge^s\Omega_m=\oplus P_nX'^S$ where S runs through all the distinct subsets $S=\{\alpha^1,\dots,\alpha^s\}$ of the set $\mathcal{H}_n(m)$ that contains s (distinct) elements and $X'^S:=x'^{\alpha^1}\wedge\cdots\wedge x'^{\alpha^s}$, the order in X'^S is fixed for each S. So, each element θ of $\wedge^s\Omega_m$ is a unique sum $\theta=\sum p_SX'^S$ where $p_S\in P_n$. For $S=\{\alpha^1,\dots,\alpha^s\}$,

$|S| := \sum_{i=1}^s |\alpha^i|$. Let us define the degree $\mathrm{Deg}(\theta)$ by the rule: $\mathrm{Deg}(0) := \infty$ and $\mathrm{Deg}(\theta) = \min\{|S| \mid p_S \neq 0\}$. For the nonzero element θ, the sum

$$\ell(\theta) := \sum \{p_S X'^S \mid |S| = \mathrm{Deg}(\theta),\ p_S \neq 0\}$$

is called the *leading term* of θ. So, $\theta = \ell(\theta) + \cdots$ where the three dots denote the *higher terms*. For all elements $\theta, \eta \in \wedge^s \Omega_m$ and $p \in P_n \backslash \{0\}$,

$$\mathrm{Deg}(p\theta) = \mathrm{Deg}(\theta) \text{ and } \mathrm{Deg}(\theta + \eta) \geq \min\{\mathrm{Deg}(\theta), \mathrm{Deg}(\eta)\}.$$

For each $j \in \mathbb{N}$, let $F^s_{\geq j} := F^s_{\geq j}(m) := \{\theta \in \wedge^s \Omega_m \mid \mathrm{Deg}(\theta) \geq j\}$. Then

$$F^s_{\geq 0}(m) = \cdots = F^s_{\geq s}(m) \supseteq F^s_{\geq s+1}(m) \supseteq \cdots \supseteq F^s_{\geq j}(m) \supseteq \cdots$$

is a descending chain of left R-modules where all but finitely many elements of the filtration are equal to zero. In this case, we say that the filtration is a *finite* filtration. Clearly, for all $i, j, s, t \geq 0$,

$$F^s_{\geq i}(m) F^t_{\geq j}(m) \subseteq F^{s+t}_{\geq i+j}(m).$$

For each $j \in \mathbb{N}$, let $F^s_j(m) := \{\theta \in \wedge^s \Omega_m \mid \mathrm{Deg}(\theta) = j\}$. Then $F^s_{\geq j}(m) = \oplus_{i \geq j} F^s_i(m)$. In particular, $\wedge^s \Omega_m = \oplus_{j \geq s} F^s_j(m)$. So, the *associated graded* left R-module,

$$\mathrm{gr}(\wedge^s \Omega_m) := \bigoplus F^s_{\geq j}(m)/F^s_{\geq j+1}(m) \simeq \bigoplus_{j \geq s} F^s_j(m) = \wedge^s \Omega_m,$$

coincides with the left R-module $\wedge^s \Omega_m$. For all $i, j, s, t \geq 0$, $F^s_i(m) F^t_j(m) \subseteq F^{s+t}_{i+j}(m)$. By (10), (where $p \in P_n$),

$$d_{m,s} : \wedge^s \Omega_m \to \wedge^{s+1} \Omega_m, \quad \theta = p x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} \mapsto d_{m,s}(\theta) \tag{25}$$

where

$$\begin{aligned} d_{m,s}(\theta) &= \sum_{0 \neq \beta \in \mathbb{N}^n} \frac{(-1)^{|\beta|+1}}{\beta!} \frac{\partial^\beta p}{\partial x^\beta}\, x'^\beta \wedge x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} + I^{m+1} \\ &= \sum_{1 \leq |\beta| \leq m-t} \frac{(-1)^{|\beta|+1}}{\beta!} \frac{\partial^\beta p}{\partial x^\beta}\, x'^\beta \wedge x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} + I^{m+1} \text{ and } t = \sum_{i=1}^s |\alpha^i|. \end{aligned}$$

It follows that

$$d_{m,s}(F^s_{\geq j}(m)) \subseteq F^{s+1}_{\geq j+1}(m). \tag{26}$$

So, the differential $d_{m,s}$ increases the degree Deg by at least 1 and we defined the *associated graded differential of graded degree* $+1$ by the rule

$$\mathrm{gr}(d_{m,s}) : \mathrm{gr}(\wedge^s \Omega_m) \to \mathrm{gr}(\wedge^{s+1} \Omega_m)$$

where for each $j \geq s$,

$$\begin{aligned} \mathrm{gr}(d_{m,s}) &: F^s_j(m) = F^s_{\geq j}(m)/F^s_{\geq j+1}(m) &&\to F^{s+1}_{j+1}(m) = F^{s+1}_{\geq j+1}(m)/F^{s+1}_{\geq j+2}(m), \\ &\qquad \theta + F^s_{\geq j+1}(m) &&\mapsto d_{m,s}(\theta) + F^{s+1}_{\geq j+2}(m). \end{aligned}$$

By (25), for $\theta = px'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} \in F_j^s(m)$ where $p \in P_n$,

$$\mathrm{gr}(d_{m,s})(\theta + F^s_{\geq j+1}(m)) = \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} x'_i \wedge x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} + F^{s+1}_{\geq j+2}(m). \tag{27}$$

The next theorem describes the cohomology groups $H^i(P_n, m)$. The key idea is to use the *finite* filtration $\{F^i_{\geq j}(m)\}$ on $\wedge^i\Omega_m$, the explicit form of $\mathrm{gr}(d_{m,i})$ (see (27)) and the fact that the representatives of the cohomology group H^i_{gr} of the associate graded cochain complex $(\mathrm{gr}(\wedge^i\Omega_m), \mathrm{gr}(d_{m,i}))$ are, in fact, cocycles of the cochain complex $(\wedge^i\Omega_m, d_{m,i})$.

Theorem 2.7 *For the polynomial algebra P_n, $\mathrm{rk}(\Omega_m) = \binom{n+m}{n} - 1$, by Proposition 2.3.(2). Let K be a field of characteristic zero. Then for all $n, m \geq 1$,*

$$H^i(P_n, m) \simeq \begin{cases} K^{\binom{\mathrm{rk}(\Omega_m)-n}{i}} & \text{if } 0 \leq i \leq \mathrm{rk}(\Omega_m) - n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By (17), $\Omega_m = \oplus_{\alpha \in \mathcal{H}_n(m)} P_n x'^{\alpha}$ and $\mathrm{rk}(\Omega_m) = |\mathcal{H}_n(m)| = \binom{n+m}{n} - 1$ is the number of free generators of the (left or right) P_n-module Ω_m. Let $e_1 := (1, 0, ..., 0), \ldots, e_n := (0, 0, ..., 1)$ and $B_n := \{e_1, \ldots, e_n\}$. Clearly, $B_n \subseteq \mathcal{H}_n(m)$ and

$$\mathcal{H}_n(m) = B_n \sqcup CB_n$$

where $CB_n := \mathcal{H}_n(m) \backslash B_n$ is the complement of the set B_n in $\mathcal{H}_n(m)$. It is obvious that

$$\wedge^{\bullet}\Omega_m = \bigoplus_{s=0}^{\mathrm{rk}(\Omega_m)} \wedge^s \Omega_m$$

where $\wedge^0\Omega_m := R$. Therefore, $H^s := H^s(P_n, m) = 0$ for all $s > \mathrm{rk}(\Omega_m)$. By (25),

$$K \subseteq \ker(d_{m,0}) \subseteq \{P \in P_n \mid \tfrac{\partial P}{\partial x_1} = \cdots = \tfrac{\partial P}{\partial x_n} = 0\} = K,$$

and so $H^0 = \ker(d_{m,0}) = K$. It remains to consider the groups H^s where $s = 1, \ldots, \mathrm{rk}(\Omega_m)$. Clearly,

$$\wedge^s\Omega_m = \bigoplus_{S \in B_n(s)} P_n X'^S \oplus \bigoplus_{S \in W_n(s)} P_n X'^S, \tag{28}$$

$$B_n(s) := B_{n,m}(s) := \{S \subseteq \mathcal{H}_n(m) \mid |S| = s \text{ and } S \cap B_n \neq \emptyset\},$$

$$W_n(s) := W_{n,m}(s) := \{S \subseteq \mathcal{H}_n(m) \mid |S| = s \text{ and } S \cap B_n = \emptyset\},$$

where for $S = \{\alpha^1, \ldots, \alpha^s\}$, $X'^S := x'^{\alpha^1} \wedge x'^{\alpha^2} \wedge \cdots \wedge x'^{\alpha^s}$ and the order of the elements in the wedge product can be arbitrary but fixed for each set S. Let $\mathcal{B}_n(s) := \bigoplus_{S \in B_n(s)} P_n X'^S$ and $\mathcal{W}_n(s) := \bigoplus_{S \in W_n(s)} P_n X'^S$. By (28),

$$\wedge^s\Omega_m = \mathcal{B}_n(s) \oplus \mathcal{W}_n(s). \tag{29}$$

The vector space $Z^s := \ker(d_{m,s})$ (resp., $B^s := \mathrm{im}(d_{m,s-1})$) admits the induced descending filtration $\{Z^s_{\geq j} := Z^s \cap F^s_{\geq j}(m)\}_{j\geq s}$ (resp., $\{B^s_{\geq j} := B^s \cap F^s_{\geq j}(m)\}_{j\geq s}$). Then

$$\mathrm{gr}(H^s) = \bigoplus_{j\geq s} H^s_j \tag{30}$$

where $H^s_j := Z^s_{\geq j}/Z^s_{\geq j}\cap(B^s+Z^s_{\geq j+1}) \simeq Z^s_{\geq j}/(Z^s_{\geq j+1}+(Z^s_{\geq j}\cap B^s)) = Z^s_{\geq j}/(Z^s_{\geq j+1}+B^s_{\geq j})$. We denote by $H^\bullet_{\mathrm{gr}} = \{H^s_{\mathrm{gr}}\}_{s\geq 0}$ the cohomology groups of the associated graded complex $(\mathrm{gr}(\wedge^\bullet\Omega_m), \mathrm{gr}(d_m))$:

$$\cdots \xrightarrow{\partial_{s-2}} \mathrm{gr}(\wedge^{s-1}\Omega_m) \xrightarrow{\partial_{s-1}} \mathrm{gr}(\wedge^{s}\Omega_m) \xrightarrow{\partial_{s}} \mathrm{gr}(\wedge^{s+1}\Omega_m) \xrightarrow{\partial_{s+1}} \cdots$$

where $\partial_s := \mathrm{gr}(d_{m,s})$. Let $Z^s_{\mathrm{gr}} := \ker(\partial_s)$, $B^s_{\mathrm{gr}} := \mathrm{im}(\partial_{s-1})$ and $H^s_{\mathrm{gr}} = Z^s_{\mathrm{gr}}/B^s_{\mathrm{gr}}$. Then $H^s_{\mathrm{gr}} = \oplus_{j\geq s} H^s_{\mathrm{gr},j}$ where

$$H^s_{\mathrm{gr},j} = \frac{\ker(F^s_{\geq j} \xrightarrow{\partial_s} F^{s+1}_{\geq j+1})}{\mathrm{im}(F^{s-1}_{\geq j-1} \xrightarrow{\partial_{s-1}} F^s_{\geq j})}.$$

Clearly, each H^s_j is a subfactor of $H^s_{\mathrm{gr},j}$ (given vector spaces $V_1 \subseteq V_2 \subseteq V$, the factor space V_2/V_1 is called a *subfactor* of V). In fact, we will see that $H^s_j = H^s_{\mathrm{gr},j}$ (see Step 6).

Step 1. $Z^s_{\mathrm{gr}} = Z^s_b \oplus Z^s_w$ *where* $Z^s_b := Z^s_{\mathrm{gr}} \cap \mathcal{B}_n(s)$ *and* $Z^s_w := Z^s_w(n,m) := Z^s_{\mathrm{gr}} \cap \mathcal{W}_n(s)$: Let $a \in Z^s_{\mathrm{gr}}$. By (29), $a = a_b + a_w$ where $a_b \in \mathcal{B}_n(s)$ and $a_w \in \mathcal{W}_n(s)$. Then $0 = \partial_s(a) = \partial_s(a_b) + \partial_s(a_w)$ implies $\partial_s(a_b) = 0$ and $\partial_s(a_w) = 0$ since, by (27),

$$\partial_s(a_b) \in \sum\{P_n X'^S \mid |S| = s+1, |S\cap B_n| \geq 2\}$$

and

$$\partial_s(a_w) \in \sum\{P_n X'^S \mid |S| = s+1, |S\cap B_n| = 1\}.$$

Therefore, $Z^s_{\mathrm{gr}} = Z^s_b \oplus Z^s_w$ as required.

Step 2. $B^s_{\mathrm{gr}} = \mathrm{im}(\partial_{s-1}) \subseteq \mathcal{B}_n(s)$: The inclusion is obvious.
By Steps 1 and 2,

$$H^s_{\mathrm{gr}} = (Z^s_b \oplus Z^s_w)/B^s_{\mathrm{gr}} \simeq Z^s_b/B^s_{\mathrm{gr}} \oplus Z^s_w.$$

Step 3. $Z^s_w = \sum_{S\in W_n(s)} KX'^S \simeq K^{|W_n(s)|}$ *and* $|W_n(s)| = \binom{|\mathcal{H}_n(m)|-n}{s}$: Let $a \in Z^s_w$, i.e., $a = \sum_{S\in W_n(s)} p_S X'^S$. By (27),

$$0 = \partial_s(a) = \sum_{S\in W_n(s)} \sum_{i=1}^n \frac{\partial p_S}{\partial x_i} x'_i \wedge X'^S.$$

Hence, $\frac{\partial p_S}{\partial x_i} = 0$ for all $i = 1, \ldots, n$, and we must have $p_S \in \bigcap_{i=1}^n \ker_{P_n}(\frac{\partial}{\partial x_i}) = K$. That is, $a \in \sum_{S \in W_n(s)} K X'^S$, as required.

Step 4. *$Z_b^s / B_{\mathrm{gr}}^s = 0$ and $H_{\mathrm{gr}}^s = Z_w^s$ for $s \geq 1$*: The main reason why this equality holds is that

$$H_{DR}^s(P_n) = 0 \text{ for } s \geq 1.$$

Let $S \in B_n(s)$. Then

$$S = S_b \sqcup S_w \text{ where } S_b := S \cap B_n \neq \emptyset \text{ and } S_w := S \cap CB_n.$$

Let $a \in Z_b^s$, i.e., $a = \sum_{S \in B_n(s)} p_S X'^S$, $p_S \in P_n$ and, by (27),

$$0 = \partial_s(a) = \sum_{S \in B_n(s)} \partial_s(p_S X'^{S_b} \wedge X'^{S_w}) =$$

$$\sum_{S \in B_n(s)} \sum_{i=1}^n \frac{\partial p_S}{\partial x_i} x_i' \wedge X'^{S_b} \wedge X'^{S_w} = \sum_{S_w} (\sum_{S_b} \sum_{i=1}^n \frac{\partial p_S}{\partial x_i} x_i' \wedge X'^{S_b}) \wedge X'^{S_w}.$$

Therefore each expression in the brackets must be equal to zero and can be written as

$$\partial_{s-|S_w|}(\sum_{S_b} p_S X'^{S_b}) = 0,$$

or, equivalently,

$$\partial_{s-|S_w|}(\sum_{T \subseteq B_n, |T| = |S| - |S_w|} p_{S = S_w \sqcup T} X'^T) = 0.$$

Since $H_{DR}^s(P_n) = 0$ for $s \geq 1$ and $|T| \geq 1$ as $S \in B_n(s)$, then Step 4 follows. Therefore, $H_{\mathrm{gr}}^s = Z_w^s$, as required.

Step 5. $d_{m,s}(Z_w^s) = 0$ (by Step 3 and (25)).

Step 6. $H_j^s = H_{\mathrm{gr},j}^s$: By Step 4, we have the equality $H_{\mathrm{gr}}^s = Z_w^s$. Hence, H_j^s is a factor vector space of $H_{\mathrm{gr},j}^s$. Now, by Step 5 and finiteness of the filtration on $\wedge^s \Omega_m$, $H_j^s = H_{\mathrm{gr},j}^s$. $\square$

When $m = 1$, Theorem 2.7 gives the classical result - the cohomology groups of the de Rham complex for the polynomial algebra.

Corollary 2.8

$$H^i(P_n, 1) \simeq \begin{cases} K & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $m = 1$, $\mathrm{rk}(\Omega_1) = \binom{n+1}{n} - 1 = n$ and $\binom{\mathrm{rk}(\Omega_1) - n}{i} = \binom{0}{i}$. Now, by Theorem 2.7, the corollary follows. $\square$

Given a cochain complex, $(C^\bullet, d)$ such that $H^i(C^\bullet) = 0$ for all but finitely many i and $\dim_K(H^i(C^\bullet)) < \infty$. The number

$$\chi(C) := \sum_i (-1)^i \dim_K H^i(C^\bullet)$$

is called the *Euler characteristic of* $C^\bullet$. The next corollary shows that the Euler characteristic of all complexes $\wedge^\bullet \Omega_m$ is 0 for $m \geq 1$.

Corollary 2.9 *For all $m \geq 1$,*

$$\sum_{i\geq 0}(-1)^i \dim_K H^i(P_n, m) = \begin{cases} 1 & m = 1, \\ 0 & m > 1. \end{cases}$$

Proof. The case $m = 1$ is obvious, see Corollary 2.8. For $m \geq 2$ and $n \geq 1$,

$$\begin{array}{rcl} r := \mathrm{rk}(\Omega_1) - 1 & = & \binom{n+m}{n} - 1 = \frac{(n+m)(n-1+m)\cdots(n-(n-1)+m)}{n!} - 1 \\ & = & (1+\frac{m}{n})(1+\frac{m}{n-1})\cdots(1+m) - 1 > m+1-1 = m \geq 2. \end{array}$$

Then

$$\sum_{i\geq 0}(-1)^i \dim_K H^i(P_n, m) = \sum_{i\geq 0}(-1)^i \binom{r}{i} = (1-1)^r = 0, \ \text{ since } r \geq 2. \ \Box$$

The next corollary gives an explicit K-basis for the vector space $H^s(P_n, m)$.

Corollary 2.10 *For all $s \geq 1$,*

$$H^s(P_n, m) = Z_w^s = \{\textstyle\sum_{S\in W_n(s)} \lambda_S X'^S \mid \lambda_S \in K\}.$$

Proof. The equalities $H^s(P_n, m) = Z_w^s$ $(s \geq 1)$ were established in the proof of Theorem 2.7.$\Box$ For each natural number $n \geq 1$ and $s \geq 1$, let

$$\mathcal{H}_n(\infty) := \cup_{m\geq 1}\mathcal{H}_n(m) = \mathbb{N}^n \backslash \{0\},$$

$$B_{n,\infty}(s) := \cup_{m\geq 1} B_{n,m}(s) = \{S \subseteq \mathbb{N}^n \mid |S| = s, \ S \cap B_n \neq \emptyset\},$$

$$W_{n,\infty}(s) := \cup_{m\geq 1} W_{n,m}(s) = \{S \subseteq \mathbb{N}^n \mid |S| = s, \ S \cap B_n = \emptyset\},$$

$$Z_w^s(n, \infty) := \{\textstyle\sum_{S\in W_{n,\infty}(s)} \lambda_S X'^S \mid \lambda_S \in K\} \simeq K^{W_{n,\infty}(s)},$$

where the sum is an infinite sum, it can be seen as a function on the set $W_{n,\infty}(s)$ taking values in K. As a vector space, $Z_w^s(n, \infty)$ is precisely the vector space of all functions from $W_{n,\infty}(s)$ to K.

Theorem 2.11 *1.*

$$\varprojlim_m H^s(P_n, m) \simeq \begin{cases} K & \text{if } s = 0, \\ K^{\mathbb{N}} & \text{if } s > 0. \end{cases}$$

2. For all $s \geq 1$, $\varprojlim_m H^s(P_n, m) \simeq Z_w^s(n, \infty)$.

Proof. 1. The case $s = 0$ is obvious as $H^0(P_n, m) = K$ and the sequence (24) for $s = 0$ is

$$\cdots \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} 0\ .$$

For $s \geq 1$, statement 1 follows from statement 2.

2. By Corollary 2.10, for all $s \geq 1$, $H^s(P_n, m) = Z_w^s(n, m)$. So, the chain (24) takes the form

$$\ldots \longrightarrow Z_w^s(n,m) \xrightarrow{\delta_m} Z_w^s(n,m-1) \longrightarrow \ldots \longrightarrow Z_w^s(n,1) = H_{DR}^s(P_n) = 0$$

where

$$\delta_m(X'^S) = \begin{cases} X'^S & \text{if } S \in W_{n,m-1}(s), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\varprojlim_m H^s(P_n, m) = Z_w^s(n, \infty)$. $\square$

3 The cohomology groups $H^i(S_n, m)$ where S_n is an algebra of power series

The aim of this section is to find the cohomology groups $H^i(S_n, m)$ where $S_n = K[[x_1, \ldots, x_n]]$ is the algebra of power series in n variables over a field K of characteristic zero (Theorem 3.2). The algebra of power series $(S_n, \mathfrak{m})$ is a local Noetherian algebra where $\mathfrak{m} = (x_1, \ldots, x_n)$ is a unique maximal ideal of S_n. The algebra S_n is a complete topological algebra with respect to the $\mathfrak{m}$-*adic* topology, i.e., $\{\mathfrak{m}^i\}_{i \geq 0}$ is the set of open neighbourhoods of 0. The tensor product of algebras $S_n \otimes S_n$ is a topological algebra where the topology τ is determined by the set $\{\mathfrak{m}^i \otimes S_n + S_n \otimes \mathfrak{m}^i\}_{i \geq 0}$ of open neighbourhoods of 0. The map $d: S_n \to S_n \otimes S_n$, $s \mapsto s' = s \otimes 1 - 1 \otimes s$ is a continuous map. In particular, by (10), for all power series $p \in S_n$,

$$p' = \sum_{\beta \neq 0} (-1)^{|\beta|+1} \frac{\partial^\beta p}{\partial x^\beta} \frac{x'^\beta}{\beta!} = \sum_{\beta \neq 0} \frac{x'^\beta}{\beta!} \frac{\partial^\beta p}{\partial x^\beta}, \tag{31}$$

where both sums are infinite sums.

Proposition 3.1 *Let $S_n := K[[x_1, ..., x_n]]$ be a power series algebra over a field K of characteristic zero. Then*

1. *$I = S_n S_n' = \oplus_{|\alpha| \geq 1} S_n x'^\alpha = S_n' S_n = \oplus_{|\alpha| \geq 1} x'^\alpha S_n$ where $\alpha \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $m \geq 1$, $I^m = \oplus_{|\alpha| \geq m} S_n x'^\alpha = \oplus_{|\alpha| \geq m} x'^\alpha S_n$. The ideal I of $S_n \otimes S_n$ is equal to $(x_1', \ldots, x_n')$.*

2. *For $m \geq 1$,*

$$\Omega_m = I/I^{m+1} = \oplus_{1 \leq |\alpha| \leq m} S_n x'^\alpha = \oplus_{1 \leq |\alpha| \leq m} x'^\alpha S_n. \tag{32}$$

   In particular, the free left/right S_n-module Ω_m has rank $rk(\Omega_m) = \binom{n+m}{n} - 1$.

3. $\mathcal{P}(S_n) = S_n[[x_1^{'}, \dots, x_n^{'}]] = [[x_1^{'}, \dots, x_n^{'}]]S_n$ *is the algebra of power series with coefficients in the algebra* S_n *and*

$$\Omega_\infty = (x_1^{'}, \dots, x_n^{'}) = \sum_{i=1}^n \mathcal{P}(S_n)x_i^{'} = \sum_{i=1}^n x_i^{'}\mathcal{P}(S_n)$$

is the ideal of the algebra $\mathcal{P}(S_n)$ *generated by the elements* $x_1^{'}, \dots, x_n^{'}$*. The derivation*
$d_\infty : R \to \Omega_\infty$ *is given by (31).*

4. *For all* $m \geq 1$*,*

$$\Omega_m = \Omega_\infty / \Omega_\infty^{m+1}. \tag{33}$$

Proof. 1. By Lemma 2.1 and Lemma 2.2, $I = S_n S_n^{'} = \sum_{|\alpha|\geq 1} S_n(x^\alpha)' = \oplus_{|\alpha|\geq 1} S_n x'^\alpha$ and $I = S_n^{'} S_n = \sum_{|\alpha|\geq 1}(x^\alpha)' S_n = \oplus_{|\alpha|\geq 1} x'^\alpha S_n$ since $(x')^\alpha = x^\alpha \otimes 1 + \cdots + 1 \otimes x^\alpha$. Hence,

$$I^m = \bigoplus_{|\alpha|\geq m} S_n x'^\alpha = \bigoplus_{|\alpha|\geq m} x'^\alpha S_n \tag{34}$$

for all $m \geq 1$. Clearly, the ideal I of the algebra $S_n \otimes S_n$ is generated by the elements $x_1', \dots, x_n'$.
2. Step 2 follows from statement 1.
3. Step 3 follows from statement 2.
4. Step 4 follows from statement 3. $\Box$

The degree Deg **and the associative filtration on** $\wedge^s \Omega_m$**.** For each $s = 1, \dots, |\mathcal{H}_n(m)|$, $\wedge^s \Omega_m = \oplus S_n X'^S$ where S runs through all the distinct subsets $S = \{\alpha^1, \dots, \alpha^s\}$ of the set $\mathcal{H}_n(m)$ that contains s (distinct) elements and $X'^S := x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s}$. So, each element θ of $\wedge^s \Omega_m$ is a unique sum $\theta = \sum p_S X'^S$ where $p_S \in S_n$. For $S = \{\alpha^1, \dots, \alpha^s\}$, $|S| := \sum_{i=1}^s |\alpha^i|$. Let us define the degree $\mathrm{Deg}(\theta)$ by the rule: $\mathrm{Deg}(0) := \infty$ and $\mathrm{Deg}(\theta) = \min\{|S| \mid p_S \neq 0\}$. For the nonzero element θ,

$$\ell(\theta) := \sum \{p_S X'^S \mid |S| = \mathrm{Deg}(\theta),\ p_S \neq 0\}$$

is called the *leading term* of θ. So, $\theta = \ell(\theta) + \cdots$ where the three dots denote the *higher terms*. For all elements $\theta, \eta \in \wedge^s \Omega_m$ and $p \in S_n \backslash \{0\}$,

$$\mathrm{Deg}(p\theta) = \mathrm{Deg}(\theta) \text{ and } \mathrm{Deg}(\theta + \eta) \geq \min\{\mathrm{Deg}(\theta), \mathrm{Deg}(\eta)\}.$$

For each $j \in \mathbb{N}$, let $F^s_{\geq j}(m) := \{\theta \in \wedge^s \Omega_m \mid \mathrm{Deg}(\theta) \geq j\}$. Then

$$F^s_{\geq 0}(m) = \cdots = F^s_{\geq s}(m) \supseteq F^s_{\geq s+1}(m) \supseteq \cdots \supseteq F^s_{\geq j}(m) \supseteq \cdots$$

is a descending chain of left R-modules where all but finitely many elements of the filtration are equal to zero. So, it is a *finite* filtration. Clearly, for all $i, j, s, t \geq 0$,

$$F^s_{\geq i}(m)F^t_{\geq j}(m) \subseteq F^{s+t}_{\geq i+j}(m).$$

For each $j \in \mathbb{N}$, let $F^s_j(m) := \{\theta \in \wedge^s\Omega_m \mid \mathrm{Deg}(\theta) = j\}$. Then $F^s_{\geq j}(m) = \oplus_{i\geq j}F^s_j(m)$. In particular, $\wedge^s\Omega_m = \oplus_{j\geq s}F^s_j(m)$. So, the *associated graded* left R-module,

$$\mathrm{gr}(\wedge^s\Omega_m) := \bigoplus F^s_{\geq j}(m)/F^s_{\geq j+1}(m) \simeq \bigoplus_{j\geq s} F^s_j(m) = \wedge^s\Omega_m,$$

coincides with the left R-module $\wedge^s\Omega_m$. For all $i,j,s,t \geq 0$, $F^s_i(m)F^t_j(m) \subseteq F^{s+t}_{i+j}(m)$. By (31), (where $p \in S_n$),

$$d_{m,s}: \wedge^s\Omega_m \to \wedge^{s+1}\Omega_m, \;\; \theta = px'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} \mapsto d_{m,s}(\theta) \tag{35}$$

where

$$\begin{aligned} d_{m,s}(\theta) &= \textstyle\sum_{0\neq\beta\in\mathbb{N}^n} \frac{(-1)^{|\beta|+1}}{\beta!}\frac{\partial^\beta p}{\partial x^\beta}\, x'^\beta \wedge x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} + I^{m+1} \\ &= \textstyle\sum_{1\leq|\beta|\leq m-t} \frac{(-1)^{|\beta|+1}}{\beta!}\frac{\partial^\beta p}{\partial x^\beta}\, x'^\beta \wedge x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} + I^{m+1} \;\text{ and } t = \sum_{i=1}^s |\alpha^i|. \end{aligned}$$

It follows that

$$d_{m,s}(F^s_{\geq j}(m)) \subseteq F^{s+1}_{\geq j+1}(m). \tag{36}$$

So, the differential $d_{m,s}$ increases the degree Deg by at least 1 and we defined the *associated graded differential of graded degree* +1 by the rule

$$\mathrm{gr}(d_{m,s}):\; \mathrm{gr}(\wedge^s\Omega_m) \to \mathrm{gr}(\wedge^{s+1}\Omega_m)$$

where for each $j \geq s$,

$$\begin{aligned} \mathrm{gr}(d_{m,s}) \;\; &: F^s_j(m) = F^s_{\geq j}(m)/F^s_{\geq j+1}(m) &&\to F^{s+1}_{j+1}(m) = F^{s+1}_{\geq j+1}(m)/F^{s+1}_{\geq j+2}(m), \\ &\quad \theta + F^s_{\geq j+1}(m) &&\mapsto d_{m,s}(\theta) + F^{s+1}_{\geq j+2}(m). \end{aligned}$$

By (35), for $\theta = px'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} \in F^s_j(m)$ where $p \in S_n$,

$$\mathrm{gr}(d_{m,s})(\theta + F^s_{\geq j+1}(m)) = \sum_{i=1}^n \frac{\partial p}{\partial x_i} x'_i \wedge x'^{\alpha^1} \wedge \cdots \wedge x'^{\alpha^s} + F^{s+1}_{\geq j+2}(m). \tag{37}$$

Theorem 3.2 describes the cohomology groups of $H^i(S_n, m)$.

Theorem 3.2 *For all* $n, m \geq 1$,

$$H^i(S_n, m) \simeq \begin{cases} K^{\binom{\mathrm{rk}(\Omega_m)-n}{i}} & \text{if } 0 \leq i \leq \mathrm{rk}(\Omega_m) - n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{rk}(\Omega_m) := \binom{n+m}{n} - 1$.

Proof. We keep the notation of the proof of Theorem 2.7. By Lemma 3.1.(2), $\Omega_m = \oplus_{\alpha\in\mathcal{H}_n(m)} S_n x'^{\alpha}$ and $\mathrm{rk}(\Omega_m) = |\mathcal{H}_n(m)| = \binom{n+m}{n} - 1$ is the number of free generators of the (left or right) S_n-module Ω_m. Notice that $\wedge^{\bullet}\Omega_m = \oplus_{s=0}^{\mathrm{rk}(\Omega_m)} \wedge^s \Omega_m$. Therefore, $H^s := H^s(S_n, m) = 0$ for all $s > \mathrm{rk}(\Omega_m)$. By (35),

$$K \subseteq \ker(d_{m,0}) \subseteq \{P \in S_n \mid \tfrac{\partial P}{\partial x_1} = \cdots = \tfrac{\partial P}{\partial x_n} = 0\} = K,$$

and so $H^0 = \ker(d_{m,0}) = K$. It remains to consider the groups H^s where $s = 1, \ldots, \mathrm{rk}(\Omega_m)$. Clearly,

$$\wedge^s\Omega_m = \bigoplus_{S\in B_n(s)} S_n X'^S \oplus \bigoplus_{S\in W_n(s)} S_n X'^S, \tag{38}$$

$$B_n(s) := B_{n,m}(s) := \{S \subseteq \mathcal{H}_n(m) \mid |S| = s \text{ and } S \cap B_n \neq \emptyset\},$$

$$W_n(s) := W_{n,m}(s) := \{S \subseteq \mathcal{H}_n(m) \mid |S| = s \text{ and } S \cap B_n = \emptyset\},$$

where for $S = \{\alpha^1, \ldots, \alpha^s\}$, $X'^S := x'^{\alpha^1} \wedge x'^{\alpha^2} \wedge \cdots \wedge x'^{\alpha^s}$ and the order of the elements in the wedge product can be arbitrary but fixed for each set S. Let $\mathcal{B}_n(s) := \bigoplus_{S\in B_n(s)} S_n X'^S$ and $\mathcal{W}_n(s) := \bigoplus_{S\in W_n(s)} S_n X'^S$. By (38),

$$\wedge^s\Omega_m = \mathcal{B}_n(s) \oplus \mathcal{W}_n(s). \tag{39}$$

The vector space $Z^s := \ker(d_{m,s})$ (resp., $B^s := \mathrm{im}(d_{m,s-1})$) admits the induced descending filtration $\{Z^s_{\geq j} := Z^s \cap F^s_{\geq j}(m)\}_{j\geq s}$ (resp., $\{B^s_{\geq j} := B^s \cap F^s_{\geq j}(m)\}_{j\geq s}$). Then

$$\mathrm{gr}(H^s) = \bigoplus_{j\geq s} H^s_j \tag{40}$$

where $H^s_j := Z^s_{\geq j}/Z^s_{\geq j}\cap(B^s + Z^s_{\geq j+1}) \simeq Z^s_{\geq j}/(Z^s_{\geq j+1} + Z^s_{\geq j}\cap B^s) = Z^s_{\geq j}/(Z^s_{\geq j+1} + B^s_{\geq j})$. We denote by $H^{\bullet}_{\mathrm{gr}} = \{H^s_{\mathrm{gr}}\}_{s\geq 0}$ the cohomology groups of the associated graded complex $(\mathrm{gr}(\wedge^{\bullet}\Omega_m), \mathrm{gr}(d_m))$:

$$\cdots \xrightarrow{\partial_{s-2}} \mathrm{gr}(\wedge^{s-1}\Omega_m) \xrightarrow{\partial_{s-1}} \mathrm{gr}(\wedge^s\Omega_m) \xrightarrow{\partial_s} \mathrm{gr}(\wedge^{s+1}\Omega_m) \xrightarrow{\partial_{s+1}} \cdots$$

where $\partial_s := \mathrm{gr}(d_{m,s})$. Let $Z^s_{\mathrm{gr}} := \ker(\partial_s)$, $B^s_{\mathrm{gr}} := \mathrm{im}(\partial_{s-1})$, and $H^s_{\mathrm{gr}} = Z^s_{\mathrm{gr}}/B^s_{\mathrm{gr}}$. Then $H^s_{\mathrm{gr}} = \oplus_{j\geq s} H^s_{\mathrm{gr},j}$ where

$$H^s_{\mathrm{gr},j} = \frac{\ker(F^s_{\geq j} \xrightarrow{\partial_s} F^{s+1}_{\geq j+1})}{\mathrm{im}(F^{s-1}_{\geq j-1} \xrightarrow{\partial_{s-1}} F^s_{\geq j})}.$$

Clearly, each H^s_j is a subfactor of $H^s_{\mathrm{gr},j}$. In fact, we will see that $H^s_j = H^s_{\mathrm{gr},j}$ (see Step 6).

Step 1. *$Z^s_{\mathrm{gr}} = Z^s_b \oplus Z^s_w$ where $Z^s_b := Z^s_{\mathrm{gr}} \cap \mathcal{B}_n(s)$ and $Z^s_w := Z^s_w(n,m) := Z^s_{\mathrm{gr}} \cap \mathcal{W}_n(s)$*: Let $a \in Z^s_{\mathrm{gr}}$. By (39), $a = a_b + a_w$ where $a_b \in \mathcal{B}_n(s)$ and $a_w \in \mathcal{W}_n(s)$. Then $0 = \partial_s(a) = \partial_s(a_b) + \partial_s(a_w)$ implies $\partial_s(a_b) = 0$ and $\partial_s(a_w) = 0$ since, by (37),

$$\partial_s(a_b) \in \sum\{S_n X'^S \mid |S| = s+1, |S \cap B_n| \geq 2\}$$

and

$$\partial_s(a_w) \in \sum\{S_n X'^S \mid |S| = s+1, |S \cap B_n| = 1\}.$$

Therefore, $Z^s_{\text{gr}} = Z^s_b \oplus Z^s_w$ as required.

Step 2. $B^s_{\text{gr}} = \text{im}(\partial_{s-1}) \subseteq \mathcal{B}_n(s)$: The inclusion is obvious.
By Steps 1 and 2,

$$H^s_{\text{gr}} = (Z^s_b \oplus Z^s_w)/B^s_{\text{gr}} \simeq Z^s_b/B^s_{\text{gr}} \oplus Z^s_w.$$

Step 3. $Z^s_w = \sum_{S\in W_n(s)} KX'^S \simeq K^{|W_n(s)|}$ *and* $|W_n(s)| = \binom{|\mathcal{H}_n(m)|-n}{s}$: Let $a \in Z^s_w$, i.e., $a = \sum_{S\in W_n(s)} p_S X'^S$. By (37),

$$0 = \partial_s(a) = \sum_{S\in W_n(s)} \sum_{i=1}^n \frac{\partial p_S}{\partial x_i} x'_i \wedge X'^S.$$

Hence $\frac{\partial p_S}{\partial x_i} = 0$ for all $i = 1, \ldots, n$, and we must have $p_S \in \bigcap_{i=1}^n \ker_{S_n}(\frac{\partial}{\partial x_i}) = K$. That is, $a \in \sum_{S\in W_n(s)} KX'^S$, as required.

Step 4. $Z^s_b/B^s_{\text{gr}} = 0$ *and* $H^s_{\text{gr}} = Z^s_w$ *for* $s \geq 1$: The main reason why this equality holds is that

$$H^s_{DR}(S_n) = 0 \text{ for } s \geq 1.$$

Let $S \in B_n(s)$. Then

$$S = S_b \sqcup S_w \text{ where } S_b := S \cap B_n \neq \emptyset \text{ and } S_w := S \cap CB_n.$$

Let $a \in Z^s_b$, i.e., $a = \sum_{S\in B_n(s)} p_S X'^S$, $p_S \in S_n$ and, by (37),

$$0 = \partial_s(a) = \sum_{S\in B_n(s)} \partial_s(p_S X'^{S_b} \wedge X'^{S_w}) =$$
$$\sum_{S\in B_n(s)} \sum_{i=1}^n \frac{\partial p_S}{\partial x_i} x'_i \wedge X'^{S_b} \wedge X'^{S_w} = \sum_{S_w} (\sum_{S_b} \sum_{i=1}^n \frac{\partial p_S}{\partial x_i} x'_i \wedge X'^{S_b}) \wedge X'^{S_w}.$$

Therefore, each expressions in the brackets must be equal to zero and can be written as

$$\partial_{s-|S_w|}(\sum_{S_b} p_S X'^{S_b}) = 0,$$

or, equivalently,

$$\partial_{s-|S_w|}(\sum_{T\subseteq B_n, |T|=|S|-|S_w|} p_{S=S_w\sqcup T} X'^T) = 0.$$

Since $H^s_{DR}(S_n) = 0$ for $s \geq 1$ and $|T| \geq 1$ as $S \in B_n(s)$, then Step 4 follows. Therefore, $H^s_{\text{gr}} = Z^s_w$, as required.

Step 5. $d_{m,s}(Z^s_w) = 0$ (by Step 3 and (35)).

Step 6. $H^s_j = H^s_{\text{gr},j}$: By Step 4 we have the equality $H^s_{\text{gr}} = Z^s_w$. Hence, H^s_j is a factor vector space of $H^s_{\text{gr},j}$. Now, by Step 5 and finiteness of the filtration on $\wedge^s \Omega_m$, $H^s_j = H^s_{\text{gr},j}$. $\square$

Corollary 3.3 *For all $m \geq 1$,*

$$\sum_{i \geq 0} (-1)^i \dim_K H^i(S_n, m) = \begin{cases} 1 & m = 1, \\ 0 & m > 1. \end{cases}$$

Proof. The case $m = 1$ is obvious, since

$$H^i(S_n, 1) \simeq \begin{cases} K & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $m \geq 2$ and $n \geq 1$,

$$\begin{array}{rcl} r := \mathrm{rk}(\Omega_1) - 1 & = & \binom{n+m}{n} - 1 = \frac{(n+m)(n-1+m)\cdots(n-(n-1)+m)}{n!} - 1 \\ & = & (1 + \frac{m}{n})(1 + \frac{m}{n-1}) \cdots (1 + m) - 1 > m + 1 - 1 = m \geq 2. \end{array}$$

Then

$$\sum_{i \geq 0} (-1)^i \dim_K H^i(S_n, m) = \sum_{i \geq 0} (-1)^i \binom{r}{i} = (1-1)^r = 0, \quad \text{since } r \geq 2. \ \Box$$

The next corollary gives an explicit K-basis for the vector space $H^s(S_n, m)$.

Corollary 3.4 *For all $s \geq 1$,*

$$H^s(S_n, m) = Z_w^s = \{\textstyle\sum_{S \in W_n(s)} \lambda_S X'^S \mid \lambda_S \in K\}.$$

Proof. The equalities $H^s(S_n, m) = Z_w^s$ $(s \geq 1)$ were established in the proof of Theorem 3.2.$\Box$

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